Measurement in classical and quantum physics

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Abstract. It was recently shown that quantum and classical mechanics are related in a deeper and more intimate way than previously thought possible. A geometric framework for both theories that allowed going back and forth between quantum and classical processes was discovered. The quantum-mechanical experiments were presented in a new and illuminating fashion, paving a way to resolve the paradoxes of quantum theory. The downside of the theory is a relatively involved machinery of functional analysis and differential geometry used to derive the results. At the same time, the main conclusions are fundamentally simple and can be presented without complicated math. The goal of this paper is to provide such a "no math" presentation of the theory.

1. Introduction

Recent papers [1]-[7] put the classical and the quantum dynamics on an equal footing opening a new way of investigating major paradoxes of quantum mechanics. The papers are based on the following three components:

(i) The standard Schrödinger quantum mechanics

(ii) The classical Newtonian mechanics

(iii) An identification of the classical space $\mathbb{R}^3$ with a submanifold in the Hilbert space of states, formed by the delta-like states of a particle

The first two components don't need a clarification. The third one is essential for the new results in the papers. Mathematically, an identification between points in $\mathbb{R}^3$ and the states of a particle located at the points is easy to construct. It is also easy to motivate this correspondence physically by noticing that to identify a point in $\mathbb{R}^3$ is to observe a particle at the point. In quantum mechanics, a particle at a point is given by the Dirac delta state, hence the identification between the two. More importantly, as proven in [1] and, in part, in the previous publications [2]-[7], the identification of the classical space and classical phase space of a system of particles with a submanifold of the space of states of the corresponding quantum system is physically sound and consistent. When the system is constrained to the submanifold, it behaves classically. Otherwise, it behaves quantum-mechanically. By fully embracing such an identification, the physical reason behind the constraint can be found and the paradoxical nature of the quantum elucidated. At the end, the analysis strongly supports the hypothesis that the appropriate arena for all physical processes is the space of states, rather than the classical space. This transition from a "point in space" to a "point in space of states" description of the physical world as well as new experimentally testable predictions are what sets the approach apart from the existing approaches and interpretations of quantum mechanics. Here the results of [1] are reviewed and explained in a simple language and put in the context of an existing research. It needs to be stressed that no additional assumptions besides (i)-(iii) above will be used in the paper.

2. Newtonian mechanics in the Hilbert space of states

Quantum systems are described most completely by state vectors that are elements of a Hilbert space. A Hilbert space is a vector space with an inner product and the norm defined by it. As such, Hilbert space is similar to the Euclidean space $\mathbb{R}^3$ that we live in with the norm being similar to the Euclidean norm (i.e., the Euclidean length of a vector). The difference is that Hilbert spaces are generally infinite-dimensional. Vectors in a Hilbert space are commonly given by complex-valued functions ψ, often called state or wave functions rather than three-component tuples of real numbers $\mathbf{x} = (x_k)$. Likewise, the norm-squared of a vector $||\psi||^2$ is the integral of the square of the modulus of the function $\int |\psi(\mathbf{x})|^2 d^3\mathbf{x}$ with respect to a measure rather than the sum of squares of the components of a vector $\sum_k (x_k)^2$ in $\mathbb{R}^3$.

To relate the classical and the quantum, we need to have a common mathematical language for both theories. If the classical mechanics is to be derived from the quantum, this language must be *functional* in nature, that is, based on the functions in a Hilbert space of states rather than points in $\mathbb{R}^3$. On the other hand, experiment demonstrates that the position of a macroscopic particle is well defined at any time. The state function of a particle with the known position is zero outside the point where the particle is located and is given in the coordinate representation by the Dirac delta-function. So the state function of a classical point particle (material point) is ideally the Dirac delta-function. The classical space $\mathbb{R}^3$ can be identified with the set of all possible positions of the particle. Consequently, the classical space is represented in the Hilbert space of states of the particle by the set M_3 of all delta functions.

There are numerous realizations of the space of states in a form of a particular Hilbert space of functions. Many of the realizations contain Dirac delta-functions. However, the typical Hilbert space $L_2(\mathbb{R}^3)$ of states of a single particle in the coordinate representation does not contain delta functions. The state of a physical particle with a well-defined position $\mathbf{a}$ in this representation is typically identified with the Gaussian function

$$\widetilde{\delta}^3_{\mathbf{a}} = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{4}} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}}. \tag{1}$$

The parameter σ controls the width of the graph of the function. The isomorphism $\omega_\sigma : \mathbf{a} \longrightarrow \widetilde{\delta}^3_{\mathbf{a}}$ identifies then the set $\mathbb{R}^3$ of points $\mathbf{a}$ with the set M_3^σ of all Gaussian functions (1). The resulting realization of $\mathbb{R}^3$ by the set M_3^σ of Gaussian functions will be sufficient for the purpose of this paper. Moreover, as shown in [1], it is equivalent to the realization of $\mathbb{R}^3$ by the set M_3 of Dirac delta functions.

The functions (1) have a unit *norm* in $L_2(\mathbb{R}^3)$: the integral of the square of any such function is equal to one. The set of all functions of norm one form a unit sphere S^{L_2} in $L_2(\mathbb{R}^3)$. This is analogous to the set of unit vectors in $\mathbb{R}^3$ forming the sphere S^2. The set M_3^σ is then a subset of the sphere S^{L_2} that is analogous to and can be visualized as a curve on S^2. The norm in the Hilbert space induces the *metric* on the sphere, so that we can measure the distance between any two points on the sphere. This is analogous to the Euclidean metric on $\mathbb{R}^3$ giving rise to the metric on the sphere S^2: to find the distance between points on the sphere, we just apply the usual measuring tape along the surface of the sphere. The metric induced on S^{L_2} is *Riemannian*, i.e., it is given by an inner product on tangent spaces. To find the inner product of two vectors in a tangent space, we simply find the real part of their inner product in the ambient Hilbert space. This is similar to the way in which the metric induced on the sphere S^2 is obtained.

For any given point φ on the sphere S^{L_2}, consider the great circle $\{\varphi\}$ through φ, formed by the states $e^{i\alpha}\varphi$ with $\alpha \in \mathbb{R}$. We will call the great circle through the point φ the phase circle for

φ. Clearly, various phase circles either coincide or do not intersect. The resulting equivalence classes of states form the *complex projective space* or the *space of physical states.* This space is important because a constant phase factor in a state does not change the probability of measurement results on the state. It is also useful to notice that for φ in M_3^σ the phase circle for φ is orthogonal to M_3^σ at φ. That is, for φ in M_3^σ the plane through the phase circle $\{\varphi\}$ is orthogonal to the tangent space to M_3^σ at φ.

By projecting points φ in S^{L_2} to their equivalence classes $\{\varphi\}$, one obtains the fibre bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$ with the circles $\{\varphi\}$ as the fibres. The complex projective space CP^{L_2} possesses an induced Riemannian metric. Namely, given two vectors tangent to S^{L_2} at φ, the components of the vectors orthogonal to the fibre $\{\varphi\}$ can be identified with vectors tangent to the projective space CP^{L_2} at the point $\{\varphi\}$. The inner product of these vectors is just the real part of the inner product of the corresponding orthogonal components in $L_2(\mathbb{R}^3)$. That yields the Riemannian metric on CP^{L_2}, called the *Fubini-Study metric.* Furthermore, the metric on the sphere S^{L_2} or alternatively on the projective space CP^{L_2} define the induced Riemannian metric on the set M_3^σ. Because at any point φ in M_3^σ, the circle $\{\varphi\}$ through φ is orthogonal to the space tangent to M_3^σ at φ, the induced Riemannian metrics on M_3^σ are both the same.

It can be shown [1] that the distance between points $\mathbf{a}$ and $\mathbf{b}$ in the Euclidean space $\mathbb{R}^3$, measured in the units 2σ, is equal to the distance between the corresponding points $\widetilde{\delta}^3_{\mathbf{a}}$ and $\widetilde{\delta}^3_{\mathbf{b}}$ in M_3^σ in the induced metric. In other words, the metric space $\mathbb{R}^3$ with the usual Euclidean metric is isomorphic (identical to) to the metric space M_3^σ with the induced Riemannian metric. Furthermore, the space M_3^σ inherits a *differentiable structure* from the like-structure on the sphere S^{L_2} or the projective space CP^{L_2} and becomes a *manifold* with a Riemannian metric or a *Riemannian manifold.* This means that differentiation of functions on M_3^σ is also defined. The map ω_σ is then an isomorphism of the Euclidean space $\mathbb{R}^3$ and the Riemannian manifold M_3^σ or an *isometric embedding* of $\mathbb{R}^3$ into the sphere S^{L_2} and the Hilbert space $L_2(\mathbb{R}^3)$. That simply means that the spaces $\mathbb{R}^3$ and M_3^σ are mathematically the same, as manifolds with a metric.

Now, the space $\mathbb{R}^3$ is a vector space: it is possible to add vectors and multiply them by numbers in the usual way, without leaving the space. On the other hand, the space M_3^σ does not inherit the vector structure of the ambient Hilbert space $L_2(\mathbb{R}^3)$. In fact, a multiple of a vector ending on S^{L_2} or a sum of two such vectors does not result in a vector with the end point on S^{L_2} in general. However, if desirable, the vector structure on M_3^σ can be introduced independently by inducing it from the space $\mathbb{R}^3$ via the isomorphism ω_σ. That is, we can define the operations of addition $\oplus$ and multiplication by a scalar $\lambda\odot$ via $\omega_\sigma(\mathbf{a}) \oplus \omega_\sigma(\mathbf{b}) = \omega_\sigma(\mathbf{a}+\mathbf{b})$ and $\lambda \odot \omega_\sigma(\mathbf{a}) = \omega_\sigma(\lambda\mathbf{a})$. If this is done, then ω_σ becomes an isomorphism of vector spaces. However, since the obtained vector structure on M_3 is not the same as the one on the Hilbert space $L_2(\mathbb{R}^3)$, the manifold M_3 with this structure is not a subspace of $L_2(\mathbb{R}^3)$.

In addition to the classical space $\mathbb{R}^3$, the Newtonian mechanics uses extensively the phase space. The phase space of a single particle in $\mathbb{R}^3$ is the space $\mathbb{R}^3 \times \mathbb{R}^3$ of all the pairs $(\mathbf{a}, \mathbf{p})$, where $\mathbf{a}$ is the position and $\mathbf{p}$ is the momentum of the particle. On the other hand, the Gaussian wave packet centered at a point $\mathbf{a}$ for a particle of mass m with group velocity $\mathbf{v} = \mathbf{p}/m$ is given in quantum mechanics by the state function

$$\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}}. \tag{2}$$

For the particle at rest, the state function $\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x})$ reduces to the functions $\widetilde{\delta}^3_{\mathbf{a}}$, previously introduced in (1). Likewise, the state functions (2) can be obtained from the Gaussian functions (1) by changing to the moving system of coordinates. Consider the subset $M_{3,3}^\sigma$ of all such state functions $\varphi_{\mathbf{a},\mathbf{p}}$ in $L_2(\mathbb{R}^3)$, defined up to a constant phase factor each. Similarly to the case of the classical Euclidean space, the map $\Omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow M^{\sigma}_{3,3}$,

$$\Omega(\mathbf{a}, \mathbf{p}) = \varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) \tag{3}$$

can be used to identify the classical phase space $\mathbb{R}^3 \times \mathbb{R}^3$ with the manifold $M^{\sigma}_{3,3}$ with the Euclidean metric induced by the map $\Omega(\mathbf{a}, \mathbf{p})$ from the metric on $L_2(\mathbb{R}^3)$.

With this construction in place, the Euclidean space $\mathbb{R}^3$ and the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ of a particle are mathematically indistinguishable from the sets M^{σ}_3 and $M^{\sigma}_{3,3}$ furnished with the provided additional structure. Because of the isometric property of the embedding ω_σ, the components of the velocity $d\varphi_t/dt$ and acceleration $d^2\varphi_t/dt^2$ of a point $\varphi_t = \widetilde{\delta}^3_{\mathbf{a(t)}}$ moving along the manifold M^{σ}_3 coincide with their Newtonian values $d\mathbf{a}/dt$ and $d^2\mathbf{a}/dt^2$. The principle of least action and the equations of motion of the Newtonian dynamics can be formulated now in terms of the new dynamical variables $\widetilde{\delta}^3_{\mathbf{a}}$ and $d\widetilde{\delta}^3_{\mathbf{a}}/dt$ with $\widetilde{\delta}^3_{\mathbf{a}} \in M^{\sigma}_3$, or a single dynamical variable $\varphi_{\mathbf{a},\mathbf{p}} \in M^{\sigma}_{3,3}$. The motion of a material point in Newtonian dynamics is represented now by a path with values in the classical space M^{σ}_3 or a path with values in the classical phase space $M^{\sigma}_{3,3}$, where both spaces are submanifolds of the sphere S^{L_2}.

A similar realization exists for classical mechanical systems consisting of any number of particles. For example, the map $\omega_\sigma \otimes \omega_\sigma : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$, $\omega_\sigma \otimes \omega_\sigma(\mathbf{a}, \mathbf{b}) = \widetilde{\delta}^3_{\mathbf{a}} \otimes \widetilde{\delta}^3_{\mathbf{b}}$ identifies the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of a two particle system with the submanifold $M_6 = \omega_\sigma \otimes \omega_\sigma(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Hilbert space $L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$ of all possible states of the pair. Projection of velocity and acceleration of the state $\varphi(t) = \delta^3_{\mathbf{a}(t)} \otimes \delta^3_{\mathbf{b}(t)}$ onto M_6 gives the Newtonian velocity and acceleration of the particles. Note also that the isomorphism $\omega_n : \mathbb{R}^3 \times ... \times \mathbb{R}^3 \longrightarrow M^{\sigma}_{3n}$, $\omega_n(\mathbf{a}_1, ..., \mathbf{a}_n) = \widetilde{\delta}^3_{\mathbf{a}_1} \otimes ... \otimes \widetilde{\delta}^3_{\mathbf{a}_n}$ allows us to interpret n-particle states in M^{σ}_{3n} as positions of n particles in the single classical space $\mathbb{R}^3$. A similar map identifies the submanifold $M^{\sigma}_{3n,3n}$ with the classical phase space of n particles. These maps allow us to think of M^{σ}_{3n} and $M^{\sigma}_{3n,3n}$ as the classical space and phase space with n particles.

To summarize, we have learned the following lesson:

Lesson 1.

Classical Newtonian mechanics of an arbitrary system of material points can be mathematically formulated in terms of the quantum state of the system, where the state is constrained to the classical space or the phase space submanifold of the Hilbert space of states.

Note that this statement is only a mathematical fact. The question remains if there is any physics behind this reformulation. Namely, we need to figure out the relationship between the Newtonian dynamics formulated in terms of the state variable and the Schrödinger dynamics of the system.

3. Newtonian dynamics as a constrained Shrödinger dynamics

The Schrödinger equation $\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{h}\varphi$ gives the velocity of the quantum state of a system as a function of time. It can be thought of as an equation for the integral curves of the vector field $h_\varphi = -\frac{i}{\hbar}\widehat{h}\varphi$ on the space of states. To solve the equation is to find the curve in the space of states that goes through the given initial point φ_0 and has tangent vectors defined by the vector field h_φ.

More generally, quantum mechanics can be formulated in terms of vector fields on the space of states in place of the linear operators. Namely, given a self-adjoint operator $\widehat{A}$ on a space of states L_2 (i.e., an observable) the associated linear vector field A_φ (a linear function on L_2 with values in L_2) is defined by

$$A_\varphi = -i\widehat{A}\varphi. \tag{4}$$

Self-adjoint operators generate unitary transformations, so that the integral curves φ_t of the associated vector fields lay on the surface of the sphere. To visualize A_φ, we can think of a tangent vector A_φ attached to the points of the sphere S^{L_2}. The commutator of observables and the commutator (Lie bracket) of the corresponding vector fields are related in a simple way:

$$[A_\varphi, B_\varphi] = [\widehat{A}, \widehat{B}]\varphi. \tag{5}$$

So the algebra of observables is realized by the algebra of linear vector fields on the sphere.

Furthermore, recall that the sphere S^{L_2} and the projective space of states possess the induced Riemannian metric. This metric can be used to find physically meaningful components of the vector field A_φ associated with an observable. Namely, the field A_φ can be decomposed into components tangent and orthogonal to the great circle $\{\varphi\}$ formed by the points $e^{i\alpha}\varphi$ (i.e., to the fibre $\{\varphi\}$ of the fibre bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$). These components have a simple physical meaning. Namely, from

$$\overline{A} \equiv (\varphi, \widehat{A}\varphi) = (-i\varphi, -i\widehat{A}\varphi), \tag{6}$$

we see that the expected value $\overline{A}$ of an observable $\widehat{A}$ in state φ is the projection of the vector A_φ onto the fibre $\{\varphi\}$. The vector $-i\widehat{A}_\perp\varphi = -i\widehat{A}\varphi - (-i\overline{A}\varphi)$ associated with the operator $\widehat{A} - \overline{A}I$ is orthogonal to the fibre $\{\varphi\}$. Accordingly, the variance

$$\Delta A^2 = (\varphi, (\widehat{A} - \overline{A}I)^2\varphi) = (\varphi, \widehat{A}_\perp^2\varphi) = (-i\widehat{A}_\perp\varphi, -i\widehat{A}_\perp\varphi) \tag{7}$$

is the norm squared of the orthogonal component $-i\widehat{A}_\perp\varphi$.

Let us apply this to the Schrödinger equation $\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{h}\varphi$. The right hand side is the vector field h_φ associated with the Hamiltonian. The decomposition of h_φ gives the Schrödinger equation in the form

$$\frac{d\varphi}{dt} = -\frac{i}{\hbar}\overline{E}\varphi - \frac{i}{\hbar}(\widehat{h} - \overline{E})\varphi = -\frac{i}{\hbar}\overline{E}\varphi - \frac{i}{\hbar}\widehat{h}_\perp\varphi. \tag{8}$$

From this and equation (7) we conclude, in particular, that the speed of state in the projective space is equal to the uncertainty of energy. Equation (8) also demonstrates that the physical state is driven by the operator $\widehat{h}_\perp$, associated with the uncertainty in energy rather than the energy itself.

Suppose that at $t = 0$, a microscopic particle is prepared in the state (2). Recall that the set of states (2), considered for all possible values of the position $\mathbf{a}$ and momentum $\mathbf{p}$, form a submanifold $M^\sigma_{3,3}$ of the space of states that is mathematically identical to the classical phase space of the particle. For any given value of $\mathbf{p}$, the lines formed by varying the components a_k of $\mathbf{a}$ form an orthogonal coordinate grid in the classical space submanifold of $M^\sigma_{3,3}$. Likewise, by fixing $\mathbf{a}$ and changing components p_k of the momentum $\mathbf{p}$, we obtain an orthogonal coordinate grid in the momentum space submanifold of $M^\sigma_{3,3}$. Let us find the components of the orthogonal part $-\frac{i}{\hbar}\widehat{h}_\perp\varphi$ of the velocity $\frac{d\varphi}{dt}$ in the basis of the unit orthogonal vectors of the coordinate grids at $t = 0$. For $\varphi = re^{i\theta}$ and an arbitrary Hamiltonian of the form $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$, a calculation of the components of velocity $\frac{d\varphi}{dt}$ along the unit orthogonal vectors $-\widehat{\frac{\partial r}{\partial a_k}}e^{i\theta}$ (i.e., the classical space component of $\frac{d\varphi}{dt}$) yields

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial a_k}}e^{i\theta}\right)\Bigg|_{t=0} = \left(\frac{dr}{dt}, -\widehat{\frac{\partial r}{\partial a_k}}\right)\Bigg|_{t=0} = \frac{v_k}{2\sigma}. \tag{9}$$

Similarly, a calculation of the components of velocity $\frac{d\varphi}{dt}$ along the unit orthogonal vectors $i\widehat{\frac{\partial\theta}{\partial p_k}}\varphi$ (momentum space component) gives

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial\theta}{\partial p_k}}\varphi\right)\Bigg|_{t=0} = \frac{mw_k\sigma}{\hbar}, \tag{10}$$

where

$$mw_k = -\left.\frac{\partial V(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}=\mathbf{x}_0} \tag{11}$$

and σ is small enough for the linear approximation of $V(\mathbf{x})$ to be valid within intervals of length σ.

The velocity $\frac{d\varphi}{dt}$ also contains a component due to the change in σ (spreading), which is orthogonal to the fibre $\{\varphi\}$ and the phase space $M^{\sigma}_{3,3}$, and is equal to

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{d\varphi}{d\sigma}}\right) = \frac{\sqrt{2}\hbar}{8\sigma^2 m}. \tag{12}$$

Calculation of the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ at $t = 0$ gives

$$\left\|\frac{d\varphi}{dt}\right\|^2 = \frac{\overline{E}^2}{\hbar^2} + \frac{\mathbf{v}_0^2}{4\sigma^2} + \frac{m^2\mathbf{w}^2\sigma^2}{\hbar^2} + \frac{\hbar^2}{32\sigma^4 m^2}, \tag{13}$$

which is the sum of squares of the found components. This completes a decomposition of the velocity of state at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M^{\sigma}_{3,3}$.

From (9) and (10), we conclude that the phase space components of the velocity of state $\frac{d\varphi}{dt} = -\frac{i}{\hbar}\widehat{h}\varphi$ assume correct classical values at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M^{\sigma}_{3,3}$. This remains true for the time dependent potentials as well. The immediate consequence of this and the linear nature of the Schrödinger equation is the following lesson:

Lesson 2.

Under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}, t)$, the state constrained to $M^{\sigma}_{3,3} \subset CP^{L_2}$ moves like a point in the phase space representing a particle in Newtonian dynamics. More generally, Newtonian dynamics of n particles is the Schrödinger dynamics of n-particle quantum system whose state is constrained to the phase-space submanifold $M^{\sigma}_{3n,3n}$ of the projective space of states of the system.

Remarks:

(i) For the states in $M^{\sigma}_{3,3}$, the velocity of state under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}, t)$ was shown to contain the classical velocity and acceleration. On the contrary, it is possible to show [1] that there exists a unique extension of the Newtonian dynamics formulated on the classical phase space $M^{\sigma}_{3,3}$ to a unitary dynamics in the Hilbert space $L_2(\mathbb{R}^3)$, satisfying formulae (9) and (10). This uniqueness property is due to the fact that the space $M^{\sigma}_{3,3}$ is *complete* in $L_2(\mathbb{R}^3)$, which means that it exhausts all directions in $L_2(\mathbb{R}^3)$. As a result, the Newtonian evolution on $M^{\sigma}_{3,3}$ has a unique "lift" to a unitary evolution on $L_2(\mathbb{R}^3)$.

(ii) Note again that the velocity and acceleration terms in (13) are orthogonal to the fibre $\{\varphi_{\mathbf{a},\mathbf{p}}\}$ of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$, showing that these Newtonian variables have to do with the motion in the projective space CP^{L_2}. The velocity of spreading is orthogonal to the fibre and to the phase space submanifold $M^{\sigma}_{3,3}$. The implication of this is that the "concentration" of state under the collapse has nothing to do with a motion in the classical space.

(iii) A replacement of observables with the associated vector fields allows one to interpret the commutators of observables (Lie bracket of vector fields) with the curvature of the sphere of states [1]. Accordingly, the algebra of observables becomes encoded into the geometry of the space of states.

As was mentioned in the Introduction, all results derived in this paper are based on the standard quantum mechanics, standard classical mechanics and the identification of points in the classical space with the delta-states of a particle located at these points. As discussed, the latter identification is certainly valid mathematically. We now see a clear indication of the physical validity of the identification, naturally leading us to the following proposition.

Proposition

We saw that physical properties (position, velocity) of a system of classical particles are ingrained into the properties of the state function. We also saw that the dynamics of macroscopic particles is contained in the Schrödinger equation for the state constrained to a 3-dimensional submanifold in the space of states, and is described by a path with values in the submanifold. We then propose that what we call the classical space is that 3-dimensional submanifold of the space of states, with the latter being the actual arena for all physical processes.

With this accepted, the remaining results in this paper become physical consequences of the standard quantum and classical mechanics (including the theory of Brownian motion) and the proposition, with no additional assumptions. Without the proposition, the results are still valid mathematically, even if their physical significance may be denied.

4. The Born rule and the normal probability distribution

Under the embedding of the classical space $\mathbb{R}^3$ into the space of states, the variable $\mathbf{a} \in \mathbb{R}^3$ is represented by the state $\tilde{\delta}^3_{\mathbf{a}}$. The set of states $\tilde{\delta}^3_{\mathbf{a}}$ form a submanifold M^{σ}_3 in the Hilbert spaces of states $L_2(\mathbb{R}^3)$, which is "twisted" in $L_2(\mathbb{R}^3)$. It belongs to the sphere S^{L_2} and goes across the dimensions of $L_2(\mathbb{R}^3)$. The distance between the states $\tilde{\delta}^3_{\mathbf{a}}$, $\tilde{\delta}^3_{\mathbf{b}}$ on the sphere S^{L_2} or in the projective space CP^{L_2} is not equal to the distance $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$ between the points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. In fact, the former distance measures the length of a shortest line between the states while the latter is obtained using the same metric, or "measuring tape", but applied along the twisted manifold M^{σ}_3. The precise relation between the two distances is given by

$$e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{4\sigma^2}} = \cos^2\theta(\tilde{\delta}^3_{\mathbf{a}}, \tilde{\delta}^3_{\mathbf{b}}), \tag{14}$$

where θ is the Fubini-Study distance between states in CP^{L_2} [1].

The relation (14) has an immediate implication onto the form of probability distributions of random variables over M^{σ}_3 and CP^{L_2}. In particular, consider the state of a particle under a measurement as a random variable φ with a certain probability distribution that depends only on the distance between the initial and the current states. Consider the probability distribution of the random variable φ constrained to M^{σ}_3. Since in this case $\varphi = \widetilde{\delta}^3_{\mathbf{a}}$, we could equally talk about the distribution of position random variable $\mathbf{a}$ for the particle. Suppose that this distribution is normal. Then the probability distribution of ψ must satisfy the Born rule for the probability of transition between states. The opposite is also true [1]. In simple words, we have the following result:

Lesson 3.

The normal distribution law on M_3^σ implies the Born rule on CP^{L_2}. Conversely, the Born rule on the space of states implies the normal distribution law on M_3^σ.

Measurements performed on a macroscopic particle satisfy generically the normal distribution law for the measured observable. This is consistent with the central limit theorem and indicates that the specific way in which the observable was measured is not important. From (14) and Lesson 3, it then follows that the Born rule is as generic on the space of states as the normal distribution law is on the classical space $\mathbb{R}^3$. However, the relationship (14) provides only a geometric part of the story. What is the physics behind the relationship of the normal probability distribution and the Born rule?

Consider a measurement of position of a particle as an example. A common way of finding the position of a macroscopic particle is to expose it to light of sufficiently short wavelength and to observe the scattered photons. Due to the unknown path of the incident photons, multiple scattering events on the particle, random change in position of the particle, etc., the process of observation can be described by the diffusion equation with the observed position of the particle experiencing Brownian motion from the starting point during the time of observation. This results in the normal distribution of observed position of the particle.

The ability to describe the process of measurement as a diffusion seems to be a general feature of measurements in the macro-world, independent of a particular measurement set-up. The averaging process making the central limit theorem applicable and leading to the normal distribution of the position random variable can be seen, for example, as the result of random hits experienced by the particle from the surrounding particles participating in the measurement. These random hits are equally likely to come from any direction, independent of the initial position of the particle, leading to Brownian motion and the validity of the diffusion equation for the probability density of the position random variable for the particle.

Now, suppose that a microscopic particle undergoes a position measurement and is exposed to a random potential that produces the Brownian motion when applied to a macroscopic particle. it can be shown [1] that the state of such a particle is equally likely to shift in any direction in the projective space of states. So the probability to find the particle in an initial state ψ in the state $\psi + \delta\psi$ depends only on the distance (but not the direction) from ψ to $\psi + \delta\psi$ in CP^{L_2}. From Lesson 3, we conclude that the probability of transition from ψ to $\psi + \delta\psi$ must be given then by the Born rule.

Let us investigate the dynamical origin of the Born rule in more detail. For this, note that in the non-relativistic quantum mechanics, the particle, and therefore its state in a single particle Hilbert space, cannot disappear or get created. The unitary property of evolution means that the state can only move along the unit sphere in the space of states $L_2(\mathbb{R}^3)$. To express this conservation of states in the case of observation of position of the particle, consider the density of states functional $\rho_t[\varphi;\psi]$. To define it, we begin with an ensemble of particles whose initial state belongs to a neighborhood of the state ψ on the sphere of states. The functional $\rho_t[\varphi;\psi]$ measures the number of states that by the time t belong to a neighborhood of a state φ in the space of states. (As shown in [1], the motion of state under a measurement can be assumed to be happening in a finite-dimensional subspace of the space of states. The density of states functional is then well defined.)

Under the realization $\omega : \mathbb{R}^3 \longrightarrow M_3^\sigma$ in section 1, the states in M_3^σ are identified with positions of particles. So the density of states functional $\rho_t[\varphi;\psi]$ must be an extension of the usual density of particles $\rho_t(\mathbf{a};\mathbf{b})$ with initial position $\mathbf{b}$ in $\mathbb{R}^3$. In other words, we must have $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$. In the case of macroscopic particles, the conservation of the number of particles is expressed in differential form by the continuity equation. For instance, if $\rho_t(\mathbf{a};\mathbf{b})$ is the density at a point $\mathbf{a} \in \mathbb{R}^3$ of an ensemble of Brownian particles with initial position near $\mathbf{b}$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ is the current density of the particles at $\mathbf{a}$, then

$$\frac{\partial \rho_t(\mathbf{a};\mathbf{b})}{\partial t} + \nabla \mathbf{j}_t(\mathbf{a};\mathbf{b}) = 0. \tag{15}$$

We will assume that $\rho_t(\mathbf{a};\mathbf{b})$ and $\mathbf{j}_t(\mathbf{a};\mathbf{b})$ are normalized per one particle, i.e., the densities are divided by the number of particles. In this case, the particle density and the probability density can be identified.

The conservation of states of an ensemble of microscopic particles is expressed by the continuity equation that follows from the Schrödinger dynamics in an arbitrary potential. This is the same equation (15) with

$$\rho_t = |\psi|^2, \quad \text{and} \quad \mathbf{j}_t = \frac{i\hbar}{2m}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi). \tag{16}$$

For the states $\psi \in M^{\sigma}_{3,3}$ we obtain

$$\mathbf{j}_t = \frac{\mathbf{p}}{m}|\psi|^2 = \mathbf{v}\rho_t. \tag{17}$$

Because the restriction of Schrödinger evolution to $M^{\sigma}_{3,3}$ is the corresponding Newtonian evolution, the function ρ_t in (17) must be the density of particles, denoted earlier by $\rho_t(\mathbf{a};\mathbf{b})$. Once again, it gives the number of particles that start on a neighborhood of $\mathbf{b}$ and by the time t reach a neighborhood of $\mathbf{a}$. The relation $\rho_t(\mathbf{a};\mathbf{b}) = \rho_t[\tilde{\delta}^3_{\mathbf{a}};\tilde{\delta}^3_{\mathbf{b}}]$ tells us that ρ_t in (16) must be then the density of states $\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi]$. It gives the number of particles initially in a state near ψ found under the measurement at time t in the state near $\tilde{\delta}^3_{\mathbf{a}}$.

From the Schrödinger equation and the fact that the Schrödinger dynamics constrained to $M^{\sigma}_{3,3}$ is equivalent to the Newtonian one, and using nothing else, we obtained the relationship between the density of states functional at a point of M^{σ}_3 and the modulus squared of the initial state function at the corresponding point of $\mathbb{R}^3$:

$$\rho_t[\tilde{\delta}^3_{\mathbf{a}};\psi] = |\psi_t(\mathbf{a})|^2. \tag{18}$$

This result in the case of a measurement can be described as follows. We are dealing with an ensemble of states initially positioned near the point ψ so that the density of states functional is concentrated at the point ψ. As the time goes by, the states undergo a random motion in accord with the Schrödinger equation with a random potential and the density of states functional "spreads out" in the space of states. The potential that generates a Brownian motion when applied to a macroscopic particle also generates a distribution of the displacement of state that is direction-independent in the space of states and satisfies the Born rule.

The relationship (18) explains the identification of $|\psi_t(\mathbf{a})|^2$ with the probability density, which is one of the postulates in quantum theory. Indeed, the probability density to find the system in a state for an ensemble of states is proportional to the value of the density of states functional on that state, which for the states in M^{σ}_3 is given by (18). So $|\psi_t(\mathbf{a})|^2$ is the probability density to find the particle near $\mathbf{a}$ simply because this quantity is the density of quantum states near the point $\tilde{\delta}^3_{\mathbf{a}}$. If there are more states near $\tilde{\delta}^3_{\mathbf{a}}$, it becomes more likely to find the state under an observation near that point.

5. Collapse of state under a measurement

We saw that, under a measurement, the state of the particle is exposed to a random potential and gets displaced in the space of states. The density of states at a point that results from this process depends only on the distance between the initial and the end states. The value of the density of states functional for the end-states on the manifold M_3^σ is given by the square of the modulus of the initial state function at the corresponding point in $\mathbb{R}^3$. It follows that the probability of transition between the initial state and the end-state satisfies the Born rule. The following lesson follows:

Lesson 4.

Collapse of the quantum state of a system can be modeled and explained by a random motion of state on the space of states under a measurement.

This result is rather unexpected and goes against the usual understanding and modeling of the collapse. The existing models utilize various ad hoc additions to the Schrödinger equation with the goal of explaining why the state under the resulting stochastic process "concentrates" in a non-unitary way to an eigenstate of the measured observable (usually, position or energy) [8]-[19]. Instead, it is argued here that under a generic measurement, an ensemble of states with an initial position near ψ "diffuses" isotropically into the space of states by a unitary Schrödinger evolution with a random potential. The random potential term in the equation is due to uncontrollable interactions of the system with the measuring device. The potential has the same characteristics as the one that can be used to model the classical measurement. Whenever a particular state in the ensemble of states under such evolution reaches a neighborhood of an eigenstate of the measured observable, we say that the "collapse" of the state has occurred. In this case, the measuring device can record the value of the measured physical quantity.

According to this scenario, the measuring device has two separate functions. On one hand, it initiates a diffusion by creating a "noise". On the other, it registers a particular location of the diffused state. For instance, the "noise" in the position measuring device could be due to a stream of photons. The device then registers the state reaching a point in M_3^σ. Note the similarity in the role of measuring devices in quantum and classical mechanics: in both cases the devices are designed to measure a particular physical quantity and inadvertently create a "noise", which contributes to a distribution of values of the measured quantity.

The difference between the measurement of the position of microscopic and a macroscopic particles is then two-fold. First, under a measurement, the state ψ of a microscopic particle is a random variable with values in the entire space of state functions CP^{L_2} and not just in the submanifold M_3^σ. Second, unless ψ is already constrained to M_3^σ (the case that would mimic the measurement of position of a macroscopic particle), to measure position is to observe the state that "diffused" enough to reach the classical space submanifold M_3^σ. Assuming the state has reached M_3^σ, the probability density of reaching a particular point in M_3^σ is given, as we saw, by the Born rule.

We don't use the term collapse of position random variable when measuring position of a macroscopic particle. Likewise, there seems to be no physics in the term collapse of the state of a microscopic particle. Instead, due to the diffusion of state, there is a probability density to find the state of the particle in various locations on CP^{L_2}. In particular, the state may reach the space manifold $M_3^\sigma = \mathbb{R}^3$. If that happens and we have detectors spread over the space, then one of them clicks. If the detector at a point $\mathbf{a}$ clicks, that means the state is at the point $\tilde{\delta}^3_{\mathbf{a}}$ in CP^{L_2} (that is, the state *is* $\tilde{\delta}^3_{\mathbf{a}}$). The number of clicks at different points $\mathbf{a}$ when experiment is repeated is given by the Born rule. The state is not a "cloud" in $\mathbb{R}^3$ that shrinks to a point under observation. Rather, the state is a point in CP^{L_2} which may or may not be on $\mathbb{R}^3 = M_3^\sigma$. When the detector clicks, we know that the state is on M_3^σ.

Note once again that there is no need for any new mechanism of "collapse" in the model. An observation is not about a "concentration" of state in $\mathbb{R}^3$ and the stochastic process initiated by the observation is in agreement with the conventional Schrödinger equation with a randomly fluctuating potential ("noise"). The origin of the potential depends on the type of measuring device or properties of the environment capable of "measuring" the system. Fluctuation of the potential can be traced back to thermal motion of molecules, atomic vibrations in solids, vibrational and rotational molecular motion, and the surrounding fields. Most importantly, the evolution under the potential happens in the *space of states*, rather than the classical space $\mathbb{R}^3$.

6. Classical behavior of macroscopic bodies

We saw that the Schrödinger evolution of state constrained to the classical phase space $M^{\sigma}_{3,3}$ results in the Newtonian motion of the particle. A similar result holds true for systems of particles. To reconcile the laws of quantum and classical physics, one must explain the nature of this constraint. Why would microscopic particles be free to leave the classical space, while macroscopic particles be bound to it?

Consider for simplicity a crystalline solid. The position of one cell in the solid defines the position of the entire solid. If one of the cells was observed at a certain point r, the state of the solid immediately after the observation (in one dimension with r being the left most cell) is the product

$$\varphi = \tilde{\delta}_r \otimes \tilde{\delta}_{r+\Delta} \otimes \ \dots \ \otimes \tilde{\delta}_{r+n\Delta}, \tag{19}$$

where Δ is the lattice length parameter. The general quantum-mechanical state of the solid is then a superposition of states (19) for different values of r in space:

$$\varphi = \sum_r C_r \tilde{\delta}_r \otimes \tilde{\delta}_{r+\Delta} \otimes \ \dots \ \otimes \tilde{\delta}_{r+n\Delta}. \tag{20}$$

Why would non-trivial superpositions of this sort be absent in nature?

To understand the dynamics of macroscopic bodies under a measurement, consider first the Brownian motion of a crystalline solid under the influence of the surrounding particles. The motion of any solid can be represented by the motion of its center of mass $\mathbf{a}$ under the total force acting on the body and a rotational motion about the center of mass. The motion of the center of mass is the motion of a material point under the random force term, which is the sum of forces acting from the surrounding particles on each cell. Therefore, the center of mass will experience the usual Brownian motion. In particular, the mean-squared displacement of the center of mass of our one-dimensional solid is given by

$$\frac{d\overline{a^2}}{dt} = 2K, \tag{21}$$

where K is the diffusion coefficient.

It is a well established and experimentally confirmed fact that macroscopic bodies experience an unavoidable interaction with the surroundings. Their "cells" are pushed in all possible directions by the surrounding particles. For instance, a typical Brownian particle of radius between $10^{-9}m$ and $10^{-7}m$ experiences about 10^{12} random collisions per second with surrounding atoms in a liquid. The number of collisions of a solid of radius $10^{-3}m$ in the same environment is then about 10^{19} per second. Collisions with photons and other surrounding particles must be also added. Even empty space has on average about 450 photons per cm^3 of space.

Let us estimate the value of the diffusion coefficient for a macroscopic body. As known after the works of Stokes and Einstein, the diffusion coefficient for a spherical particle is well described by the expression

$$K = \frac{k_B T}{6\pi\eta r}, \tag{22}$$

where r us the radius of the particle and η is the dynamic viscosity. In particular, for a macroscopic particle of radius $r \sim 1mm$ in the air, $\eta \sim 10^{-5} N \cdot s/m^2$, at room temperature, we get $K \sim 10^{-12}m^2/s$. According to (21), it would take about $10^6 s$ or more than 10 days for the standard deviation of $1mm$ in the distribution of the displacement of the particle to occur. Now, the actual time of observation of position of particles in experiments is much shorter. For instance, if we scatter visible light off the particle to determine its position, the time interval of observation could be as short as $10^{-13}s$, which for a $1mm$ of radius particle in the air would amount to the displacement of the order of $10^{-21}m$. This quantity is much less than the accuracy of measurement, limited by the wavelength $\lambda \sim 10^{-5}m$, and cannot be observed in the measurement. In general, the vanishingly-small value of the diffusion coefficient K for macroscopic bodies together with the zero mean displacement explains why the Brownian motion is not commonly observed in the macro-world.

Consider now what happens to the quantum state of a macroscopic body under the influence of the same surroundings. As in section 4, we conclude that the state of the solid will experience a random motion on the space CP^{L_2}, built on the superpositions (20) and that any direction of displacement of the state in the space at any time t is equally likely. From (14) it follows that for the close-by states in M_3^{σ} we have the equality $\theta^2 = a^2$. That is, the Fubini-Study distance between the states and the Euclidean distance between the corresponding points in $\mathbb{R}^3$ are the same. From this, the isotropy of the distribution of states and (21) it then follows that at $t = 0$

$$\frac{d\overline{\theta^2}}{dt} = 2K, \tag{23}$$

where the coefficient K is the same as in (21). As discussed, the value of K for macroscopic bodies is vanishingly small. Accordingly, the position of the state ψ in the space of states for a macroscopic body remains constant. As a rule of thumb, if the Brownian motion of the body can be neglected, then the change in the quantum state of the body under the influence of the surrounding can be neglected as well.

The considered situation is surprisingly similar to that of a pollen grain and a ship initially at rest in still water. While under the kicks from the molecules of water, the pollen grain experiences a Brownian motion, the ship in still water will not move at all. We see that this is more than an analogy: when the state is constrained to the classical phase space submanifold, the "pushes" experienced by the state become the classical kicks in the space that could lead to the Brownian motion of the body.

As an example, let us consider again the particle of $1mm$ radius in the air at room temperature whose displacement during the time of observation was estimated earlier by $10^{-21}m$. The Fubini-Study distance between Gaussian states in M_3^{σ} that are $10^{-21}m$ apart with $\sigma \sim 10^{-5}m$ can be calculated via (14) and is about $10^{-16}rad$. So the state is hardly moving away from its original position and cannot realistically reach points in the space of states that are away from that position. In particular, it becomes impossible to find the state positioned initially in the configuration space M_{3n}^{σ} at a different point in the space of states.

Suppose now an external potential V is applied to the macroscopic system. According to (13), this will "push" the state that belongs to the classical phase space submanifold in the direction tangent to the submanifold. Therefore, the external potential applied to a macroscopic body will not affect the motion of state in the directions orthogonal to the classical phase space submanifold. That means that the state will remain constrained to the submanifold. On the other hand, as we know from the same section, the constrained state will evolve in accord with Newtonian dynamics in the total potential $V + V_S$, where V_S is the potential created by the surroundings. However, since at any time t the total force $-\nabla V_S$ exerted on the macroscopic body by the particles of the surroundings can typically be neglected (no friction), the body will evolve according to Newtonian equations with the force term $-\nabla V$.

So, the origin of the classical behavior of macroscopic bodies in the theory is two-fold. First of all, the initial state of a macroscopic body is on $M_{3,3}^{\sigma}$. That is, a macro body is created at a point of the submanifold $M^{\sigma}_{3,3}$. Second, because of the interaction of the particle with the surroundings (radiation, molecules of air, water and other media), the state undergoes a diffusion process rather than a free Schrödinger evolution. Also, because of the macroscopic character of the body, the diffusion coefficient is extremely small. The probability distribution of the variation of the state of the body has a zero mean and is nearly constant in time. We don't see a quantum evolution of the state, but rather a negligible diffusion. This diffusion does not influence measurement of position of the body as that measurement happens on a much shorter time scale.

From this analysis, it becomes clear that the transition of macroscopic to microscopic happens for the macroscopic bodies for which the Brownian motion in the surrounding media is observable. If a macroscopic body is sufficiently small so that the Brownian motion of the body in the media can be observed in an experiment, then the superposition of states of different positions of the body becomes observable as well. In fact, it was demonstrated that under the conditions typical for the Brownian motion, the state of the system has equal probability of any direction of displacement in the space of states. In particular, the state may become a superposition of distinguishable states of a given position in $\mathbb{R}^3$. Interference effects on such states are then observable.

7. The double-slit experiment

The derivation of Newtonian from Schrödinger dynamics, the relationship of the Born rule to the normal probability distribution, a clear picture of collapse and an explanation of the classical behavior of macroscopic bodies all support the hypothesis in section 2 stating that the isomorphism between the classical space $\mathbb{R}^3$ and the manifold M^{σ}_3 must be considered a physical and not just a mathematical identification. In the remaining part of the paper this hypothesis combined with the standard classical and quantum mechanics will be used to analyze quantum-mechanical experiments and to address the paradoxes of quantum theory. As discussed, the superposition principle in quantum mechanics represents the main obstacle to reconciliation of the quantum and the classical. Let us therefore begin with the simplest manifestation of the superposition principle: the double-slit experiment.

Different forms of the experiment are well known and don't need to be reviewed here. We are going to discuss the simplest set-up of the experiment, involving an electron gun, a plate with a pair of parallel slits, and a scintillating screen or a photographic plate to observe the interference pattern. We will deal with a single electron. Also, since the origin of the electron will not be important, the electron gun will be left out of the picture. For now we will also leave out the screen registering the outgoing particles and the surroundings.

The Hilbert space of the system is the tensor product of spaces $L_2(\mathbb{R}^3)$, one for each particle in the system. However, the state of the macroscopic plate with the slits has the form (19) in section 6. That is, the plate is given by a point ψ_P on the submanifold $M^{\sigma}_{3n} = M^{\sigma}_3 \otimes ... \otimes M^{\sigma}_3$ in CP^{L_2}. Here L_2 is the tensor product of Hilbert spaces $L_2(\mathbb{R}^3)$ for all particles of the plate. As discussed in the section 2, the isomorphism $\omega_n : \mathbb{R}^3 \times ... \times \mathbb{R}^3 \longrightarrow M^{\sigma}_{3n}$, $\omega_n(\mathbf{a}_1, ..., \mathbf{a}_n) = \tilde{\delta}^3_{\mathbf{a}_1} \otimes ... \otimes \tilde{\delta}^3_{\mathbf{a}_n}$ allows us to view the states in M^{σ}_{3n} as points in the classical configuration space $\mathbb{R}^{3n}$ or positions of n particles in the single classical space $\mathbb{R}^3$. That is how our usual view of the plate becomes possible and how the state ψ_P gets identified with a set of material points that represent the particles of the plate in $\mathbb{R}^3$. Although it is useful to "visualize" the plate by the state ψ_P, or by the corresponding set of points in $\mathbb{R}^3$, the effect of the plate on the electron in the experiment can be more easily described by a potential $\widehat{V}$, which is infinite on the plate and zero at the slits.

We can now proceed with the analysis of the experiment. First, the wave packet of the electron propagates toward the plate. If the electron is sufficiently fast, the spreading of the packet on the approach to the plate can be neglected. During this time interval, the propagation of the initial packet ψ is happening essentially by a displacement $\psi_t(\mathbf{x}) = \psi(\mathbf{a}-\mathbf{v}t)$. The electron state moves along (parallel to) the classical space submanifold M_3^σ in $L_2(\mathbb{R}^3)$. The motion can be thought of in the classical terms; we have a material point propagating towards the plate.

During the second stage of the experiment, the electron goes "through" the slits in the plate. The Schrödinger evolution of the electron is still described by a path ψ_t in the Hilbert space $L_2(\mathbb{R}^3)$. However, at this time, the shape of the function ψ_t is different. After interaction with the potential the state function is a superposition $c_1\psi_1+c_2\psi_2$, where the packets ψ_1, ψ_2 represent the state of the electron passing though one of the slits with the second slit closed. The resulting superposition continues propagating in the same direction, forming a path ψ_t.

What happens at this step is very important. Let us describe the motion of the state in terms of the Schrödinger evolution on the space of states $L_2(\mathbb{R}^3)$. The state ψ_t of the electron propagates along the classical space submanifold $M_3^\sigma = \mathbb{R}^3$, approaching the area of the non-vanishing potential associated with the plate. On interaction with the potential, the state ψ_t evolves into a superposition $c_1\psi_{1t} + c_2\psi_{2t}$. In terms of the geometry on the space of states, the path ψ_t is no longer valued in the classical space submanifold M_3^σ in $L_2(\mathbb{R}^3)$. In fact, the classical space submanifold is formed by the Gaussian states. Those states have a single "hump", while ψ_t behind the plate is a "double-humped" state function. Therefore, the state at time t does *not* represent a point in the classical space. As the result of interaction with the plate, the path ψ_t moves away from the classical space M_3^σ and, therefore, passes *over* the plate with the slits (which can be thought of as a subset P of M_3^σ).

The origin of the paradox of the double-slit experiment is now clear. When trying to view the dynamics of the electron in the experiment within the classical space $M_3^\sigma = \mathbb{R}^3$, we are facing the dilemma: which slit did the electron go through? When formulated in these terms, the only correct answer seems to be that it went "through both" or to admit that position is not defined. This violently clashes with everything we know about the world around us and contradicts Newtonian mechanics. It forces us to think of the electron in terms of some kind of "electron cloud" that can "assemble" back to the particle (collapse) when measured. Alternatively, that the answer to Einstein's question - "is the moon there, when nobody looks?", - must be negative, at least for the electrons.

Under the Schrödinger dynamics, the evolution of the electron is a path ψ_t in the Hilbert space. It is a path in the usual sense; a continuous, single-valued function of time with values in $L_2(\mathbb{R}^3)$. When the state is constrained to M_3^σ, ψ_t is the usual path of a particle in Newtonian dynamics. When the electron interacts with the plate, the path continues into the Hilbert space. Because the path can be written now as a sum $\psi_t = c_1\psi_{1t} + c_2\psi_{2t}$, we tend to think that both parts, ψ_{1t} and ψ_{2t} are real, so that the path of the electron splits into the paths that go through slits 1 and 2. This is paradoxical and contradictory. In fact, if the same wave function is written as a superposition of eigenstates of a different observable, then, by the same logic, the new components must be real as well. Since there are many observables, the notion of reality becomes ill-defined. The way out is to accept that the adequate way to describe the reality is by the vector ψ_t and not by its components ψ_{1t} and ψ_{2t}, that depend on the choice of a basis.

The issue of reality of the components ψ_{1t} and ψ_{2t} is similar to the following question in classical physics. When a physical vector (say, a velocity vector) is written in terms of its components in a certain basis, should we count the components as real? The answer is obvious: the physical vector itself is real because it is basis independent. However, the components of the vector are just shadows of the real thing as they change with the change of basis, similar to the way a shadow changes when the source of light is moved around. Our problem with the superposition principle is rooted in the desire to attach to the classical components like ψ_{1t} and ψ_{2t} the status of a "real thing". The paradox of the superposition is resolved by accepting the total state ψ_t as an adequate description of reality, while considering ψ_{1t} and ψ_{2t} for what they really are: representation dependent components of the vector ψ_t. To answer Einstein's question: The moon and the electron *are* there, when nobody looks. Their existence is described by the state, at any time and not just when the object is measured. Whenever the state belongs to the classical space $\mathbb{R}^3 = M_3^\sigma$, it describes the usual classical existence in the Newtonian sense. But unlike the classical position, the state also catches the quantum origin of nature.

Suppose that position of the electron is measured by the screen behind the plate. As discussed in sections 4 and 5, a measurement of position produces a diffusion on the space of states. If the initial state of the electron was ψ, the density of states functional at the point $\tilde{\delta}_{\mathbf{a}}^3$ was shown to be $|\psi(\mathbf{a})|^2$. Because the state is a superposition of two states that describe the electron passing through one of the slits, the density of states functional contains the cross term. This term in the density results in an alternating probability of reaching different parts of the screen, producing a typical interference picture on the screen.

What happens when we place a source of light between the plate with the slits and the screen? In this case, the diffusion of the electron state begins earlier. After passing through the plate, the electron state is "two-humped". In particular, this initial state of the electron is positioned away from M_3^σ. Suppose that on interaction with the photons of the source of light, the electron is observed near one of the slits. That means, in particular, that the diffused electron state is on the classical space submanifold M_3^σ. So the state function of the electron observed near one of the slits must be "single-humped". The electron in such a Gaussian-like state is later observed on the screen. Clearly, no interference picture would appear on the screen.

What about a delayed-choice version of the experiment when we decide to determine which slit the electron went through *after* the electron has passed the plate with the slits? For instance, we could turn the light on after the electron went through the slits. The paradox is that the electron seems to "decide" retroactively to behave as a particle or a wave, and, accordingly, to go through one slit, or both, depending on our decision to turn the light on. However, the previous analysis is not altered by this change in the experiment. Whether or not the light source is present, the state of the electron after the slits is "two-humped". In particular, inserting a screen between the plate and the light source will show the interference pattern. When the light source is turned on and the electron is observed near one of the slits, the "two-humped" state is transformed to a "single-humped, Gaussian-like state. As a result, the screen behind the light source will not show the interference picture.

As before, we see that the paradox is due to our assumption that the electron must be *on* the classical space manifold M_3^σ at any time. In this case, the observed interference pattern signifies that the electron somehow "spreads out" over both slits and behaves like a wave. On the other hand, if the light source is on, then the electron visibly goes through one of the slits only and behaves like a particle. The paradox is resolved by accepting that evolution of the electron is described by a path φ_t in the space of states CP^{L_2}. When the electron interacts with the plate, the path abandons the classical space submanifold M_3^σ in CP^{L_2}, the state function is "two-humped" and the interference picture is observable. When the source of light is turned on and the electron is observed by one of the slits, the path returns to the classical space, the state function is "single-humped" and the interference is not present. The moment when the light source is turned on is irrelevant. The nature of the electron does not change. In particular, the electron does not go back in time to "adjust" its nature depending on our decision to turn the light source on. The electron does not spread over the slits. Moreover, the electron *does not go through* the slits. If anything, it goes *over* the slits into the large dimensions of the space of states and *comes back* whenever its position is measured.

8. EPR and Shrödinger cat paradoxes

Consider a pair of microscopic particles. The quantum state of the pair is an element of the tensor product Hilbert space $H = L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$. When positions of both particles are known, the state belongs to the submanifold $M_3^\sigma \otimes M_3^\sigma$ in the projective space CP^H for H. Suppose that the state of the pair is prepared to be a superposition of states $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$ for different values of a. Here we switched for simplicity to particles in one dimension. If position of one of the particles was found to be a, then position of the second is guaranteed to be $a+d$ and vice versa. Likewise, if momentum of the first particle in a pair is found to be p, then the momentum of the second will be $-p$. A pair of particles in such an entangled state is an example of EPR pair.

There are essentially two paradoxes associated with the considered pair. The first one consists of the non-local character of "communication" between the particles of the pair. Namely, how could a measurement performed on one particle instantaneously affect the other particle, no matter how far away? The other paradox is related to our ability to influence the reality of position or momentum of the second particle by choosing to measure either position or momentum of the first. This calls into question the notion of physical reality as well as completeness of quantum theory.

As in the single particle case, the evolution of the pair is a path in the space of states CP^H. Whenever the path takes values in the submanifold $M_3^\sigma \otimes M_3^\sigma$, the position of both particles is known. Moreover, if the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, then the Schrödinger dynamics of the pair is equivalent to the Newtonian one. As before, the constructed isomorphisms $\omega_\sigma \otimes \omega_\sigma$ allow us to identify the state of the pair in $M_3^\sigma \otimes M_3^\sigma$ with a point in the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of the system of two point-particles or positions of the particles in the classical space $\mathbb{R}^3$.

Suppose the state of the pair is a point on CP^H away from the submanifold $M_3^\sigma \otimes M_3^\sigma$. Suppose that position of one of the particles is measured. For instance, we could shine light onto one of the particles. Analogously to the process discribed in section 4, the measurement will trigger a random motion of the state of the pair in the space of states. The conditional probability for the state of reaching a particular point in the manifold $M_3^\sigma \otimes M_3^\sigma$, given that it has reached the manifold, satisfies the Born rule. We stress that position of only one of the particles needs to be measured for the state to be able to reach the manifold $M_3^\sigma \otimes M_3^\sigma$. Under a successful measurement, the state of the pair will undergo a random motion while following a continuous path ψ_t from the initial state to a point in $M_3^\sigma \otimes M_3^\sigma$.

It is important that the distance d between the points a and $a+d$ has nothing to do with the motion of the state ψ to an observed position state $\tilde{\delta}_a \otimes \tilde{\delta}_{a+d}$. The observed properties of one particle are not communicated to the other one by any signal or a field in space. Moreover, there are no particles in the sense of objects on $M_3^\sigma \otimes M_3^\sigma$, or on $M_3^\sigma = \mathbb{R}^3$. Rather, there is a state ψ_t representing the pair. When the state is constrained to $M_3^\sigma \otimes M_3^\sigma$, the particles are described by the classical Newtonian dynamics. So we can think of them in purely classical terms, as indeed, material points. However, the state in CP^H, not constrained to the classical space or phase space submanifolds describes the pair as a quantum object that embraces and supersedes the material point of Newtonian mechanics.

The paradox of "creation" of reality of position or momentum of one particle by measuring the corresponding quantity of the second clears up as well. These physical characteristics only make sense for the state constrained to the manifold $M_6^\sigma \otimes M_6^\sigma$ or alike. In that particular case, their relation to the motion of state was derived in section 2. Otherwise, these physical characteristics are only "shadows" of the deeper physics described by the state. The space of states is the new physical arena that extends the classical space. The state offers a more complete way of identifying characteristics of physical bodies. It generalizes the notion of position, momentum and other observed quantities and reproduces these quantities when constrained to an appropriate classical submanifold.

Consider now a system consisting of a microscopic and a macroscopic particles, initially in the product state $\varphi \otimes \tilde{\delta}^3_{\mathbf{r}}$, where φ is the initial state of the microscopic particle and $\tilde{\delta}^3_{\mathbf{r}}$ is the state of the macroscopic particle. For instance, the macroscopic particle could represent the apparatus in a measuring experiment or the cat in the Shrödinger cat experiment. The paradox here is that the existence of entangled states of microscopic systems results in a contradiction when applied to macroscopic objects. In particular, in the famous Schrödinger thought experiment we get superpositions of states of a cat being alive and dead.

As discussed in section 6, a macroscopic system is subjected to interaction with the environment, or, to put it differently, is "measured by the environment". As we know from section 6, under the influence of the environment the state of the macroscopic particle in the projective space remains constant. The macroscopic object is therefore constrained to the classical space M_3^σ. The action of any external potential on the particle is described by the Newtonian dynamics. In particular, if a potential is applied to the entire system, the macroscopic particle will change in a classical way while the state of the microscopic particle will evolve by the Schrödinger equation and the state of the system will remain the product of the new states. It follows that the entangled state of the cat and the atom is not possible. There cannot be Schrödinger cats running around.

Remarks:

(i) Note that the decoherence theory approach to the measurement of a microscopic system by a classical apparatus cannot be valid. In fact, a measurement related decoherence requires an entanglement of the measured microscopic system and the classical apparatus as a first step, which, as we know, is not possible.

(ii) Note also that the inconsistent view of reality by different observers in the Wigner's friend type of experiment discovered by Frauchiger and Renner [21] is only present when an entanglement of microscopic and macroscopic objects is possible, which is not the case. At the same time, there is much more to be investigated now that the new physical arena is potentially the entire space of states. In particular, the notion of reality is altered for the objects not constrained to the classical space submanifold. We need to understand what it means in detail.

9. Summary, experimental verification and comparison to the existing approaches

From the standard Schrödinger and Newtonian mechanics combined with an observation that points of the classical space can be identified with the delta-like states of a particle, a tight new relationship between classical and quantum physics was derived. The classical space and classical phase space for a system of particles were identified with submanifolds in the space of states. The dynamics of a classical mechanical system was identified with the Schrödinger dynamics of the system with the state constrained to the classical phase space submanifold. The Newtonian dynamics reigns on the submanifold, while the Schrödinger dynamics is its unique extension to the entire space of states. The normal probability distribution on the classical space has a unique extension to the space of states and becomes the Born rule for the probability of transition between states. Vector fields on the classical space have a unique extension to linear vector fields on the space of states. Quantum observables are identified with the associated linear vector fields. Commutators of observables are Lie brackets of vector fields and are related to the curvature of the space of states. The physical quantities of velocity, acceleration and mass of particles in Newtonian dynamics are now components of the velocity of quantum state.

The process of measurement in quantum mechanics is now an extension to the space of states of the measurement in classical physics with its typical normal distribution of the measured observable. The state under a measurement is equally likely to fluctuate in any direction of the space of states. This fact together with the geometry of embedding of the classical space into the space of states are responsible for the validity of the Born rule for the probability of transition between arbitrary states. The Born rule also follows from the relationship of the continuity equations in the Newtonian and Schrödinger dynamics resulting from the embedding. The state is not a cloud in the classical space that somehow "shrinks" under a measurement.

Rather, the state is a point in the space of states that undergoes a random motion and has a chance of reaching certain areas of the space in the process. The evolution remains unitary and satisfies the Schrödinger equation with a random potential. The "collapse" of the state becomes an unnecessary and redundant concept. The measuring device is not responsible for creating a basis into which the state is to be expanded. If several measuring devices are present, they are not "fighting" for the basis. When the eigen-manifolds of the corresponding observables do not overlap, only one of them can "click" for the measured particle as the state can reach only one eigen-manifold at a time.

The deterministic and the stochastic Schrödinger evolutions have to be clearly distinguished. The motion of state normally follows the deterministic Schrödinger equation with a given potential. However, under the conditions typically associated with a measurement, the state evolves by the Schrödinger equation with a random potential. The potential initiates a random motion of the state on the space of states and the resulting change in the density of states functional. The difference between these two types of evolution is analogous to the difference between the usual Newtonian motion of a macroscopic particle in a given potential and the Brownian motion of the particle under random hits, particularly in modeling a measurement by the diffusion.

The resulting approach to measurement is applicable to quantum systems consisting of an arbitrary number of particles. When the system is a macroscopic particle, the diffusion of state trivializes and the state remains unchanged in time. As a result, macroscopic particles are constrained to the classical space submanifold of the space of states. On the other hand, microscopic particles can leave the submanifold and exist in a superposition of position eigenstates. The double-slit and numerous other quantum-mechanical experiments demonstrate this property. When position of a microscopic particle is measured and the result is obtained, the state returns to the classical space submanifold. A particular point in the submanifold where the state was found determines the value of the position variable.

The entangled state of a system of two or more particles is represented by a point that does not belong to the classical space or phase space submanifold of the space of states. As in the case of a single particle under a measurement, the state undergoes a random motion in the space of states. An entangled pair is analogous to a pair of macroscopic particles, say, in one dimension, with a weightless rigid rod connecting them. When position of one particle in the pair is measured, the position of the second is then fixed. Similar to this, to make a measurement on an EPR-pair, it suffices to measure just one of the particles. As in the case of a single particle, the measurement yields an isotropic distribution of the displacement of state in the space of states, implying the validity of the Born rule. Even though a measurement on one of the particles in an entangled pair restricts the outcomes of the corresponding measurement on the second particle, the measurement does not imply a "communication" between the particles. This too can be understood using the example of the pair connected by a rod: A random uncontrollable motion created by measurement of position of one of the particles cannot be used to transmit information to the second. Instead, similarly to the case of a single microscopic particle, the state of on an EPR-pair moves under a measurement in a random and continuous way and has a chance of reaching an eigenstate of the measured observable.

The obtained realization of the Newtonian mechanics in functional terms and the derived relationship of the classical and quantum theories is not just a reformulation of the theory. The results of the classical and quantum mechanics are indeed reproduced in the realization. However, the embedding resulted in a tighter relationship between the theories. Not only the Newtonian dynamics is the Schrödinger dynamics with a constraint, but the Schrödinger dynamics is a *unique* dynamics with this property. This allows us to approach the process of measurement in quantum theory in a new way, as an extension of the random motion associated with a classical measurement. An important consequence of this is the notion of a density of state functional and its derived isotropy property that can be tested. In particular, if several observables are measured on a certain state of a system at the same time, we should be able to test the isotropy of the distribution of frequencies of the measured eigenvalues. That is, the state should be seen "collapsing" equally frequently to the eigenstates of different observables, positioned at the same Fubini-Study distance from the initial state. The observation of different components of spin of a particle at the same time would probably be the easiest way to set up such an experiment. Another experiment could test the classical to quantum boundary. This boundary is predicted by the theory to be determined by the largest particles for which the Brownian motion in an appropriate media is observable. In fact, as long as the Brownian motion for the particle is observable, the state of the particle is capable of diffusing into the space of states. In particular, superpositions of the position eigenstates for the particle become possible and can be observed.

The ultimate difference of the proposed realization of the classical and quantum theory from the existing approaches and interpretations of quantum mechanics is in representing the classical space as a submanifold of the space of states and in recognizing the need for extension of the arena for physical processes from the classical space to the space of states. By blindly accepting that all physics happens in the classical 3-dimensional space, we are bound to a total fiasco in understanding superpositions of classically meaningful states and the transition from quantum to classical.

The relationship of the proposed realization to the existing approaches can be spelt in more detail.

(i) In accepting the standard quantum mechanics, the realization is closest to the orthodox interpretation and could be called a completion of the latter. In the realization, the state of a quantum system is defined at any time and is given by the state function. The classical characteristics typically show up under a measurement, but they represent simply a subset of possible states in the space of states. Most importantly, we get an explanation of collapse, without needing to modify the Schrödinger equation.

(ii) The realization demonstrates that objective collapse theories are redundant. Although the idea of a diffusion-like behavior of the state is valid, we see now that there is no need for a modification of the Schrödinger equation. The evolution is stochastic, but unitary. The modification of the equation with a change in the measured observable characteristic for the collapse models is not needed. In fact, the state does not need to be driven exclusively to the eigenstates of the observable. There is no preferred basis to deal with either. The proposed mechanism explains the collapse of the state of a microscopic system under a measurement as well as the classical properties of macroscopic bodies.

(iii) The proposed realization proves that the "many worlds" interpretation of quantum theory must be wrong as there are no superpositions of position states of macroscopic objects, for instance.

(iv) The De Broglie-Bohm theory insists that all particles have a well defined position in the classical space at any time. The theory combines the classical position with the state-dependent "pilot" wave and non-local "quantum" potential to reinterpret the Schrödinger equation. But is a non-local potential any better than a non-local state function, when we have to deal additionally with the increased number of dynamical quantities (positions of particles plus the pilot wave itself)? This becomes unnecessary and redundant in the realization because of the embedding of classical space into the space of states and the derivation of Newtonian dynamics from the Schrödinger one.

(v) Interpretations of quantum mechanics based on the statistical meaning of the wave function make sense for the process of measurement on a quantum system. In this case, the process of diffusion of state is in fact described statistically by starting with an ensemble of states, as in the theory of Brownian motion. However, denying the significance of state of a single system is unwarranted and makes understanding of the Schrödinger evolution, the superpositions of states and the transition to classicality only more cumbersome if not impossible.

The obtained results lead one to the conclusion that macroscopic and microscopic bodies may not be so different after all. The only important distinction is that microscopic systems live in the space of states while their macroscopic counterparts live in the classical space submanifold of thereof. Because our life happens in the macro-world and we deal primarily with macroscopic bodies, it is hard for us to understand the infinite-dimensional quantum world around us. As soon as the classical-space-centered point of view is extended to its state-space-centered counterpart, the new, clearer view of the quantum theory and the classical-quantum relationship emerges.

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